CALCULUS 223

Final Exam Study Sheet

$\boxed{\textbf{Derivatives and Partials}}$

$$\nabla f\,(x, y, z) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

Maxima/Minima

i) f has a **local maximum** at (a, b) if $f_{xx}, f_{yy} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b);
ii) f has a **local minimum** at (a, b) if $f_{xx}, f_{yy} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b);
iii) f has a **saddle point** at $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b);
iv) The test is *inconclusive* at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). Use different method.

Tangent Plane at point p when:

$$\nabla f|_p \cdot \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = 0$$

LaGrange Muliplier

$$\nabla f = \lambda \nabla g \text{ and } g(x, y, z) = 0$$

$\boxed{\textbf{Multiple Integrals}}$

$$A_R = \int\int_R r\,dr\,d\theta = \int\int_R dy\,dx$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin(\phi)\,dr\,d\phi\,d\theta$$

$$\int_0^{2\pi}\int_{r_i}^{r_o}\int_{z_a}^{z_b} r\,dz\,dr\,d\theta$$

$$\int_{y_a}^{y_b}\int_{x_a}^{x_b}\int_{z_a}^{z_b} dz\,dx\,dy$$

Polar

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Spherical

$$x = \rho\sin(\phi)\cos(\theta) \qquad y = \rho\cos(\phi)\sin(\theta) \qquad z = \rho\cos(\phi)$$

$$\rho^2 = x^2 + y^2 + z^2$$

Change of Variables x, y to u, v

$$J(u,v) = \left\| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right\| = \frac{\partial(x,y)}{\partial(u,v)}$$

Mass and Moment

Density $= \delta(x,y)$

Mass $M = \int\int \delta(x,y)\,dA$

First Moments

$$M_x = \int\int y\delta(x,y)\,dA \qquad M_y = \int\int x\delta(x,y)\,dA$$

Center of Mass

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

Moments of Inertia (second moments)

About the x axis $I_x = \int\int y^2\delta(x,y)\,dA$

About the y axis $I_y = \int\int x^2\delta(x,y)\,dA$

About the origin $I_0 = \int\int \left(x^2 + y^2\right)\delta(x,y)\,dA = I_x + I_y$

About line L $I_L = \int\int r^2(x,y)\delta(x,y)\,dA$ where $r(x,y)$ is the distance from L to (x,y)

Radii of Gyration

About the x axis $R_x = \sqrt{\frac{I_x}{M}}$

About the y axis $R_y = \sqrt{\frac{I_y}{M}}$

About the origin $R_0 = \sqrt{\frac{I_0}{M}}$

Line Integrals

Line Integral *Find mass of string.*

$$\int F(s)\,ds = \int F(s)\sqrt{(x')^2 + (y')^2 + (z')^2}\,dt$$

Circulation *Work through vector field.*

$$\oint M\,dx + N\,dy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dx\,dy = \int \vec{F}\cdot\,d\vec{r} = \int \vec{F}\cdot\vec{T}\,ds$$

$$d\vec{r} = \nabla\vec{r}\,dt$$

$$F = \begin{pmatrix} M \\ N \end{pmatrix}$$

Circulation Density

$$\textbf{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Flux

$$\oint M\,dy - N\,dx = \int \left(M\frac{dy}{dt} - N\frac{dx}{dt}\right)\,dt = \int\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dx\,dy = \int_a^b \vec{F}\cdot\left(\frac{\partial y}{\partial t}\vec{i} - \frac{\partial x}{\partial t}\vec{j}\right)\,dt$$

Flux Density

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Green's Theorem

$$\oint_C M\,dy - N\,dx = \int\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dx\,dy$$

Green's Theorem Area Formula

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx$$

First Order Differential Equations

Seperable *Form* $P(x)dx = Q(y)dy$

1. Integrate.

Exact *Form* $M dx + N dy = 0$ when $M_y = N_x$

1. $\phi = \int M\,dx = f + h(y)$
2. $h'(y) = N - \phi_y$ because $\phi_y = f_y + h'(y)$
3. $h(y) = \int h'(y)\,dy$
4. $\phi = f + h(y)$

Linear *Form* $y' + p(x)y = q(x)$

1. $m(x) = e^{\int p(x)\,dx}$
2. $\int (my)'\,dx = \int m(x)q(x)\,dx$
3. $y = \frac{f+C}{m(x)}$

Second Order Differential Equations

Homogeneous Linear *Form* $y'' + ay' + b = 0$

1. $r^2 + ar + b = 0$ find r.
2. r_1, r_2 Real, and $r_1 \neq r_2$ $\quad y = C_1e^{r_1x} + C_2e^{r_2x}$
3. r_1, r_2 Real, and $r_1 = r_2$ $\quad y = (C_1x + C_2)e^{r_1x}$
4. r_1, r_2 Complex Conjugates, $\alpha \pm \beta i$ $\quad y = e^{\alpha x}(C_1\cos(\beta x) + C_2\sin(\beta x))$

Nonhomogeneous Linear *Form* $y'' + ay' + b = f(x)$

Method 1

1. Find y_h.
2. Find y_p in same form as f.
    a. C try A or Ax
    b. x^2 try $Ax^2 + Bx + C$
    c. $\cos(2x)$ try $A\cos(2x) + B\sin(2x)$
3. Get $y_p = f$
4. $y = y_h + y_p$

Method 2

1. Find y_h.
2. $y_p = v_1 y_1 + v_2 y_2$ $\qquad$ $v_1(x)$, $v_2(x)$ $\qquad$ (match form)
2. Solve $v_1' y_1 + v_2' y_2 = 0$ and $v_1' y_1' + v_2' y_2' = f(x)$ for v_1, v_2.
3. $y = y_h + y_p$
